

Limits

Definition We write

$$\lim_{x \rightarrow a} f(x) = L$$

and say “the limit of $f(x)$ as x approaches a , equals L ” if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a .

Definition We write

$$\lim_{x \rightarrow a^-} f(x) = L$$

and say the “left-hand limit of $f(x)$ as x approaches a is equal to L ” if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and $x < a$.

Similarly we define $\lim_{x \rightarrow a^+} f(x) = L$.

THEOREM The following is true $\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$.

LIMIT LAWS Suppose that c is a constant and the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then

1. $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2. $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3. $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
4. $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
5. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ if $\lim_{x \rightarrow a} g(x) \neq 0$
6. $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n$ where n is a positive integer
7. $\lim_{x \rightarrow a} c = c$
8. $\lim_{x \rightarrow a} x = a$
9. $\lim_{x \rightarrow a} x^n = a^n$ where n is a positive integer
10. $\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer (moreover, if n is even, we assume that $a > 0$)
11. $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$ where n is a positive integer (moreover if n is even, we assume that $\lim_{x \rightarrow a} f(x) > 0$)

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a).$$

THEOREM If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a , then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

THE SQUEEZE THEOREM If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L.$$